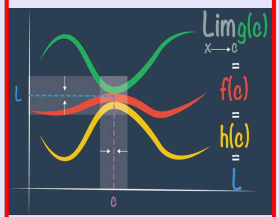
Calculus I

Lecture 5



Feb 19-8:47 AM

Class QZ 4

Box Your Final Ans

Use the definition of $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

to find $f'(x)$ for $f(x)=3x^2-6x-10$, then

Solve $f'(x)=0$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{3(x+h)^2-6(x+h)-10-3x^2+6x+10}{h}$$

$$=\lim_{h\to 0}\frac{3(x^2+2xh+h^2)-6(x+h)-10-3x^2+6x+10}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{6x}-6h\cancel{-10}\cancel{-3x^2}\cancel{+6x}\cancel{+10}}{h}$$

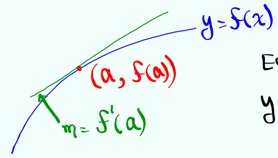
$$=\lim_{h\to 0}\frac{\cancel{h}(6x+3h-6)}{\cancel{h}}=\lim_{h\to 0}(6x+3h-6)=\boxed{6x-6}$$

Solve $f'(x)=0$

$6x-6=0$

$6x=6 \rightarrow \boxed{x=1}$

Jun 24-7:04 AM

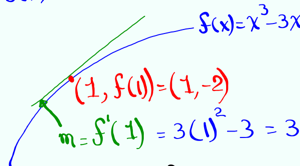


Eqn of tan. line

$y - f(a) = m(x-a)$

find eqn of the tan. line to the graph of $f(x)= x^3-3x$ at $x=1$.

$f(1)=1^3-3(1)=-2$



$m=f'(1) = 3(1)^2-3 = 3-3=\boxed{0}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3-3(x+h)-x^3+3x}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{-3x}-3h\cancel{-x^3}\cancel{+3x}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2-3)}{\cancel{h}}$$

$$=\lim_{h\to 0}(3x^2\cancel{+3xh}+\cancel{h^2}-3)=\boxed{3x^2-3}$$

$$\boxed{\begin{array}{l} y-f(a)=m(x-a)\\ y-(-2)=0(x-1)\\ y+2=0 \to \boxed{y=-2}\end{array}}$$

Jun 24-8:21 AM

Given $f(x)=x^2$

Evaluate $$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

$(x+h)^2=$
$(x+h)(x+h)=$
$x^2+xh+xh+h^2=$
$x^2+2xh+h^2$

$$\lim_{h\to 0}\frac{(x+h)^2-2x^2+(x-h)^2}{h^2}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}\cancel{+2xh}+h^2\cancel{-2x^2}\cancel{+x^2}\cancel{-2xh}+h^2}{h^2}=\lim_{h\to 0}\frac{2\cancel{h^2}}{\cancel{h^2}}$$

$$=\lim_{h\to 0}2=\boxed{2}$$

Jun 24-8:32 AM

Given $f(x) = x^3$

Evaluate $\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 2x^3 + (x-h)^3}{h^2}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + \cancel{3x^2h} + 3xh^2 + \cancel{h^3} - \cancel{2x^3} + \cancel{x^3} - \cancel{3x^2h} + 3xh^2 - \cancel{h^3}}{h^2}$$

$$= \lim_{h \to 0} \frac{6x\cancel{h^2}}{\cancel{h^2}} = \lim_{h \to 0} 6x = \boxed{6x}$$

$$(x-h)^3 = (x-h)(x-h)(x-h)$$
$$= (x-h)(x^2 - xh - hx + h^2)$$
$$= (x-h)(x^2 - 2xh + h^2)$$
$$= x^3 - 2x^2h + xh^2 - x^2h + 2xh^2 - h^3$$
$$= x^3 - 3x^2h + 3xh^2 - h^3$$

Jun 24-8:41 AM

Known Formula

$$\boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1} \quad \& \quad \boxed{\lim_{h \to 0} \frac{1 - \cos h}{h} = 0}$$

Evaluate $\lim_{h \to 0} \frac{\sin 2h}{\boxed{3}h} = \frac{1}{3} \lim_{h \to 0} \frac{\boxed{2}\sin 2h}{2h}$

$$= \frac{2}{3} \boxed{\lim_{h \to 0} \frac{\sin 2h}{2h}} = \frac{2}{3} \cdot 1 = \boxed{\frac{2}{3}}$$

Jun 24-9:03 AM

Evaluate $\lim_{x\to 1} \frac{1-\cos(x-1)}{x-1} = \frac{1-\cos(1-1)}{1-1}$

$= \frac{1-\cos 0}{0} = \frac{1-1}{0} = \frac{0}{0}$

I.F.

Let $h = x-1$

$x \to 1$

$h \to 0$

$\lim_{h\to 0} \frac{1-\cos h}{h} = \boxed{0}$

Evaluate $\lim_{x\to 1} \frac{\sin(x-1)}{x^2-1} = \frac{\sin(1-1)}{1^2-1} = \frac{\sin 0}{1-1} = \frac{0}{0}$

I.F.

$\lim_{x\to 1} \frac{\sin(x-1)}{(x+1)(x-1)} = \lim_{x\to 1}\left[\frac{1}{x+1} \cdot \frac{\sin(x-1)}{x-1}\right]$

$\boxed{\lim_{x\to 1} \frac{1}{x+1}} \cdot \boxed{\lim_{x\to 1} \frac{\sin(x-1)}{x-1}} = \frac{1}{1+1} \cdot \boxed{\lim_{h\to 0} \frac{\sin h}{h}}$

$h = x-1$

$x \to 1$

$h \to 0$

$= \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}$

Jun 24-9:08 AM

Evaluate $\lim_{h\to 0} \frac{\sin 3h}{\sin 5h} = \lim_{h\to 0} \frac{\frac{3\sin 3h}{3h}}{\frac{5\sin 5h}{5h}}$

$= \frac{3}{5} \lim_{h\to 0} \frac{\frac{\sin 3h}{3h}}{\frac{\sin 5h}{5h}}$

$= \frac{3}{5} \cdot \frac{\boxed{\lim_{h\to 0} \frac{\sin 3h}{3h}}}{\boxed{\lim_{h\to 0} \frac{\sin 5h}{5h}}}$

$= \frac{3}{5} \cdot \frac{1}{1} = \boxed{\frac{3}{5}}$

$h = .001$

$\frac{\sin(.003)}{\sin(.005)} = .6000016$

$h = .0001$

$\frac{\sin(.0003)}{\sin(.0005)} = .600000016$

It appears to reach $.6 = \frac{3}{5}$

Jun 24-9:17 AM

For $\varepsilon = .5$, find $\delta > 0$ such that $\lim_{x \to 3}(2x+5) = 11$. ✓

$f(x) = 2x+5$ $a = 3$ $L = 11$ ✓

Verify the limit ✓

$\lim_{x \to 3}(2x+5) = 2(3)+5 = 6+5 = 11$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|2x+5-11| < \varepsilon$ " $|x-3| < \delta$

$|2x-6| < \varepsilon$

$|2(x-3)| < \varepsilon$

$2|x-3| < \varepsilon \rightarrow |x-3| < \frac{\varepsilon}{2}$

$\delta = \frac{\varepsilon}{2}$

For $\varepsilon = .5$

$\delta = \frac{.5}{2} = \boxed{.25}$

Jun 24-9:44 AM

For $\boxed{\varepsilon = .1}$, find $\delta > 0$ such that $\lim_{x \to -3}(x^2+3x) = 0$. ✓

$f(x) = x^2+3x$ $a = -3$ $L = 0$ ✓

$\lim_{x \to -3}(x^2+3x) = (-3)^2+3(-3) = 9-9 = \boxed{0}$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$|x^2+3x-0| < \varepsilon$ " $|x-(-3)| < \delta$

$|x^2+3x| < \varepsilon$ " $|x+3| < \delta$

$|x(x+3)| < \varepsilon$ $\delta = \frac{\varepsilon}{c}$

$|x||x+3| < \varepsilon$

Bound keep

If $|x| < C$, then $C|x+3| < \varepsilon$, $|x+3| < \frac{\varepsilon}{C}$

Let $\delta < 1$

$|x+3| < 1$

$-1 < x+3 < 1$

Subtract 3

$-4 < x < -2$

$-4 < x < -2 < 4$

$-4 < x < 4$

$|x| < 4$

$C = 4$

$\delta = \min\left\{1, \frac{\varepsilon}{4}\right\}$

$= \min\left\{1, \frac{.1}{4}\right\}$

$= \min\left\{1, \frac{1}{40}\right\} = \boxed{\frac{1}{40}}$

$-4 < x < -2$

$\cancel{2 < x < -2}$ $-4 < x < 4$ ✓

Jun 24-9:50 AM

For $\boxed{\varepsilon = 2}$, find a $\delta > 0$ such that

$\lim_{x \to 2} x^3 = 8$ ✓

$f(x) = x^3$
$a = 2$
$L = 8$ ✓

$\lim_{x \to 2} x^3 = 2^3 = 8$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 - 8| < \varepsilon$ 〃 $|x - 2| < \delta$

$|(x-2)(x^2+2x+4)| < \varepsilon$ whenever $|x-2| < \delta$

Recall $A^3 - B^3$ factoring $\rightarrow (A-B)(A^2+AB+B^2)$

$|x-2|\,|x^2+2x+4| < \varepsilon$

$\underbrace{|x-2|}_{\text{Keep}}$ $\underbrace{|x^2+2x+4|}_{\text{Bound}}$

If $|x^2+2x+4| < C$, then $C|x-2| < \varepsilon \rightarrow |x-2| < \frac{\varepsilon}{C}$

for now $\delta = \frac{\varepsilon}{C}$

Since $f(x)$ is a polynomial function, $\delta < 1$

$|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3 \rightarrow |x| < 3$

$x^2 + 2x + 4 < 9 + 2(3) + 4$

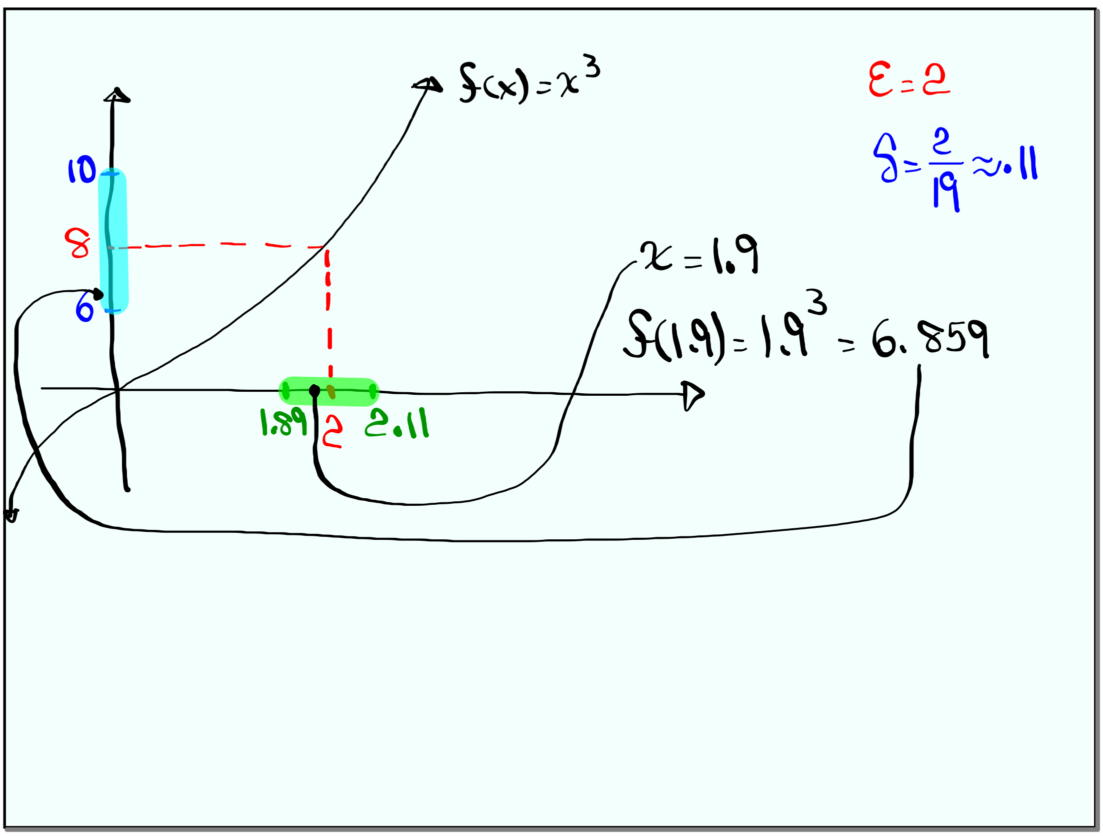
$x^2 + 2x + 4 < 19$

$|x^2+2x+4| < 19$ C

$\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$

$\delta = \min\left\{1, \frac{2}{19}\right\}$
$= 2/19 \approx .11$

Jun 24-10:02 AM



Jun 24-10:17 AM

For $\boxed{\varepsilon = .5}$, find $\delta > 0$ such that $\lim_{x \to 0} \sqrt[3]{x} = 0.$ ✓

$f(x) = \sqrt[3]{x}$ $a = 0$ $L = 0$ ✓

$\lim_{x \to 0} \sqrt[3]{x} = \sqrt[3]{0} = 0$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|\sqrt[3]{x} - 0| < \varepsilon$ 〃 $|x - 0| < \delta$

$|\sqrt[3]{x}| < \varepsilon$ 〃 $|x| < \delta$

Raise both sides to 3rd power

$|\sqrt[3]{x}|^3 < \varepsilon^3 \longrightarrow |x| < \varepsilon^3$

Pick $\delta = \varepsilon^3$

$\delta = .5^3 = \boxed{.125}$

Jun 24-10:21 AM

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2.$ ✓

$f(x) = \frac{1}{x}$, $L = 2$ ✓, $a = \frac{1}{2}$

Verify the limit ✓

$\lim_{x \to 1/2} \frac{1}{x} = \frac{1}{1/2} = 1 \div \frac{1}{2} = 1 \cdot \frac{2}{1} = 2$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{x} - 2\right| < \varepsilon$ 〃 $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{1 - 2x}{x}\right| < \varepsilon$ 〃 $\left|x - \frac{1}{2}\right| < \delta$

$\left|\frac{-2x + 1}{x}\right| < \varepsilon$

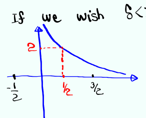
$\left|\frac{-2(x - \frac{1}{2})}{x}\right| < \varepsilon$ For now $\delta = \frac{\varepsilon}{C}$

Bound Keep

$\frac{2}{|x|}\left|x - \frac{1}{2}\right| < \varepsilon$

If $\frac{2}{|x|} < C$, then $\left|x - \frac{1}{2}\right| < \frac{\varepsilon}{C}$

If we wish $\delta < 1$,



$\left|x - \frac{1}{2}\right| < 1$

$-1 < x - \frac{1}{2} < 1$

Add $\frac{1}{2}$

$-\frac{1}{2} < x < \frac{3}{2}$

we have a Problem we need to Pick δ to be less than $1/2$, maybe $1/4$, $1/5$, $\frac{1}{10}$.

Pick $\delta < \frac{1}{4}$

$\left|x - \frac{1}{2}\right| < \frac{1}{4}$

$-\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}$

multiply by 4

$-1 < 4x - 2 < 1$

Add 2

$1 < 4x < 3$

Divide by 4

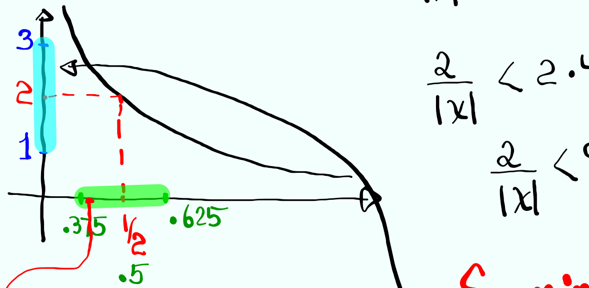
$\frac{1}{4} < x < \frac{3}{4}$

Take reciprocal

$4 > \frac{1}{x} > \frac{4}{3} \rightarrow \frac{4}{3} < \frac{1}{x} < 4$

Jun 23-10:14 AM

$\frac{4}{3} < \frac{1}{x} < 4 \rightarrow \frac{1}{|x|} < 4$



$\frac{2}{|x|} < 2 \cdot 4$

$\frac{2}{|x|} < 8$ ← C

$\delta = \min\left\{\frac{1}{4}, \frac{\varepsilon}{8}\right\}$

what is your ε?

$\varepsilon = 1$

$\delta = \min\left\{\frac{1}{4}, \frac{1}{8}\right\} = \frac{1}{8}$

$= .125$

Let $x = .376$

$f(.376) = \frac{1}{.376} = 2.659574468$

Google IVT Intermediate Value Theorem

Jun 24-10:37 AM

Find K such that the function below

$$f(x) = \begin{cases} Kx^3 - 5 & \text{if } x < 2 \\ Kx^2 + 10 & \text{if } x \geq 2 \end{cases}$$

1) $f(2) = K(2)^2 + 10$
$= 4K + 10$

is $\underbrace{\text{continuous at } x = 2.}$

$\lim_{x \to 2} f(x) = f(2)$ ✓

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$K(2)^3 - 5 = K(2)^2 + 10$

$8K - 5 = 4K + 10$

$8K - 4K = 10 + 5$

$4K = 15$ → $\boxed{K = \frac{15}{4}}$

$\boxed{K = 3.75}$

Jun 24-10:45 AM

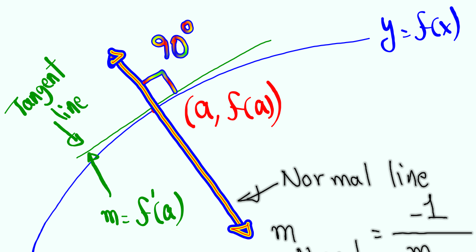
Given $x^2 < f(x) < \cos(x-1)$

Find $\lim_{x\to 1} f(x)$.

$$\lim_{x\to 1} x^2 = 1^2 = 1 < \lim_{x\to 1} f(x) < \lim_{x\to 1} \cos(x-1) = \cos 0 = 1$$

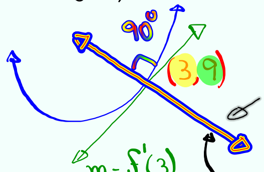
$$\lim_{x\to 1} f(x) = 1 \quad \text{by S.T.}$$

Jun 24-10:51 AM



$$m_{\text{Normal Line}} = \frac{-1}{m_{\text{Tan. Line}}} = \frac{-1}{f'(a)}$$

Find eqn of the normal line to the graph of $f(x) = x^2$ at $x = 3$.



$$m = \frac{-1}{f'(3)}$$

$$m = \frac{-1}{2(3)} = \frac{-1}{6}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

$$= 2x$$

$$y - f(a) = m(x - a)$$

$$y - 9 = \frac{-1}{6}(x - 3)$$

$$y = \frac{-1}{6}x + \frac{1}{2} + 9$$

$$\boxed{y = \frac{-1}{6}x + \frac{19}{2}}$$

Jun 24-10:55 AM